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ORIENTATION OF NATURAL TRIHEDRAL OF THE SPIRAL-HELIX SUPPORTING TRAJECTORY OF SPATIAL VEHICLE MOVEMENT

Purpose. To improve both reliability and engineering accuracy of 3D modeling of a vehicle movement.

Methodology. Orts of natural trihedral of spiral-helix supporting trajectory are expressed vectorially by means of time derivatives of a radius vector of a program movement. Quaternion matrices represent the complicated operations of vector algebra. Spatial orientation or program rotation of a natural trihedral is described with the help of Gibbs vector, Rodriguez-Hamilton vector, matrix of direction cosines depending upon kinematic parameters of the program movement.

Findings. A hodograph of program transfer of a vehicle is represented in the class of spiral-helix supporting trajectories in terms of earth reference. Unit vectors of a natural trihedral of the spiral-helix supporting trajectory have been obtained depending on the time derivatives of the program motion hodograph.

Originality. The program transfer – radius vector hodograph and program rotation (orientation) – Gibbs vector or quaternion on Rodriguez-Hamilton parameters are represented vectorially making it possible to model spatial orientation problems and problems of control of dynamic systems (i.e. vehicles) in the form of quaternion matrices.

Practical value. Calculation formulas are represented in the ordered, compacted matrix form adapted directly to computer technology. The algorithm helps solve a wide range of problems of dynamic design of vehicles.

Keywords: *hodograph, spiral-helix trajectory, natural trihedral, orientation, Gibbs vector, Rodriguez-Hamilton parameters, quaternion matrices*

Statement of the problem. It is required to develop components of control, orientation, and stabilization system within inertial space of dynamical frame of reference connected with a vehicle.

Analysis of the recent research and publications. Determination of accurate trajectory of a vehicle movement is a topical problem [1, 2].

Traditionally, spatial orientation is determined with the help of three independent rotations (i. e. Euler-Krylov angles, aircraft angles, ship angles and others) to which a number of matrices of directional cosines correspond [3, 4].

Universalization of spatial rotation description results in the necessity to use Rodriguez-Hamilton parameters and Gibbs vector. Gibbs vector makes it possible to apply mathematical apparatus of vector analysis for the statement and solution of problems of orientation, angular stabilization, and control of dynamic systems [5].

Natural trihedral rotation has been determined in Rodriguez-Hamilton parameters and Gibbs vector components expressed by means of time derivatives of a vehicle program transfer of hodograph. Description of the rotation and spatial transfer is performed on the basis of vector algebra; corresponding calculation algorithms are developed in quaternion matrices [5].

Unsolved aspects of the problem. It is required to represent vectorially program transfer – radius-vector hodograph, and program rotation (orientation) – Gibbs vector or quaternion on Rodriguez-Hamilton parameters. This makes it possible to model spatial problems of orientation and control of dynamic systems in quaternion matrices.

Objective of the paper is to determine natural trihedral of spiral-helix supporting trajectory of a program vehicle movement within the inertial space of orientation.

Description of the method. A hodograph of a program vehicle transfer according to spiral-helix supporting trajectories is proposed. Orts of natural trihedral spiral-helix supporting trajectory are expressed vectorially by means of time derivatives of radius-vector of the program movement. Quaternion matrices represent complicated operations of vector algebra. Spatial orientation or program rotation of natural trihedral is described with the help of Gibbs vector, Rodriguez-Hamilton parameters, and a matrix of directional cosines depending upon kinematic parameters of program movement. Rodriguez-Hamilton parameters and Gibbs vector components are determined on the components of a matrix of transformation of orts of the natural trihedral of the supporting trajectory and earth reference orts. From the viewpoint of the applied vector approach to the description of program transfer and rotation, it seems appropriate to model spatial problems of orientation and dynamic system (vehicle) control basing upon mathematical apparatus of monomial (1, 0, -1) – matrices (4 × 4) [5].

Presentation of the main research. Program transfer of a vehicle as a material point within the inertial space from the preset initial phase state to the final one is performed according to radius-vector hodograph of the type [5]

$$\bar{r}(t)=\left\|\rho_0\rho_1\rho_2\rho_3\right\|\left\|\begin{matrix}1\\ t\\ t^2\\ t^3\end{matrix}\right\|\left(\bar{i}\cos\omega t+\bar{j}\sin\omega t\right)+\bar{k}\left\|h_0h_1h_2h_3\right\|\left\|\begin{matrix}1\\ t\\ t^2\\ t^3\end{matrix}\right\|,$$

where $\rho_i h_i$ ($i = 0, 1, 2, 3$) are running parameters determined on the given boundary conditions; ω is the average angular rotational velocity; $\bar{i}, \bar{j}, \bar{k}$ are the basis of Cartesian reference system, and t is current time of the vehicle movement (an independent variable).

The hodograph makes it possible to model the program transfer of a material point as that being adequate and implementable in the context of motion modes of vehicles.

The method for developing a hodograph of a vehicle program movement within the considered class of spiral-helix path is to determine the introduced variable parameters according to the required boundary conditions. Auxiliary conditions also involve limitations on technical operability of the path shape geometry and a vehicle motion mode in the context of the path.

An example of the development of a hodograph of a vehicle program movement in the context of spiral-helix path. Boundary conditions within the considered path section (Fig. 1) are assumed to be set as follows

$$\begin{aligned} &t=0; && t=t_k;\\ &\bar{r}_A=\bar{i}r_{1A}+\bar{k}r_{3A}; && \bar{r}_B=\bar{j}r_{2B}+\bar{k}r_{3B};\\ &\bar{V}_A=\bar{i}V_{1A}+\bar{j}V_{2A}+\bar{k}V_{3A}; && \bar{V}_B=-\bar{i}V_{1B}+\bar{j}V_{2B}+\bar{k}V_{3B}. \end{aligned}$$

In this context $r_{2A} = 0$, $r_{1B} = 0$ and $\omega t_k = \dfrac{\pi}{2}$; i.e. the rotation is performed at a right angle.

According to the set boundary conditions, we have for A point ($t = 0$)

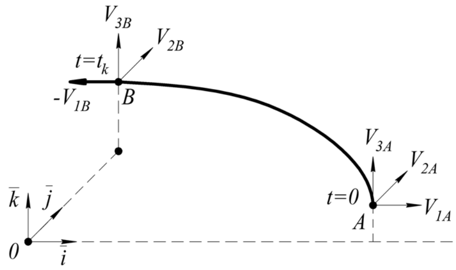

Fig. 1. Boundary conditions in terms of ascending (descending) motion while rotating

$$\begin{aligned} &r_{1A}(0)=\rho_0; && r_{2A}(0)=0; && r_{3A}(0)=h_0;\\ &\dot{r}_{1A}(0)=\rho; && \dot{r}_{2A}(0)=\omega\rho_0; && \dot{r}_{3A}(0)=h_1, \end{aligned}$$

and for B point ($t = t_k$)

$$r_{1B}(t_k)=0;$$

$$r_{2B}(t_k)=\rho_0+\rho_1 t_k+\rho_2 t_k^2+\rho_3 t_k^3;$$

$$r_{3B}(t_k)=h_0+h_1 t_k+h_2 t_k^2+h_3 t_k^3;$$

$$\dot{r}_{1B}(t_k)=-\omega\left(\rho_0+\rho_1 t_k+\rho_2 t_k^2+\rho_3 t_k^3\right);$$

$$\dot{r}_{2B}(t_k)=\rho_1+2\rho_2 t_k+3\rho_3 t_k^2;$$

$$\dot{r}_{3B}(t_k)=h_1+2h_2 t_k+3h_3 t_k^2.$$

This implies

$$\rho_0=r_{1A};\quad \rho_1=V_{1A};\quad \omega r_{1A}=V_{2A};\quad h_0=r_{3A};\quad h_1=V_{3A};$$

$$\rho_0+\rho_1 t_k+\rho_2 t_k^2+\rho_3 t_k^3=r_{2B};\quad h_0+h_1 t_k+h_2 t_k^2+h_3 t_k^3=r_{3B};$$

$$\omega\left(\rho_0+\rho_1 t_k+\rho_2 t_k^2+\rho_3 t_k^3\right)=V_{1B};\quad h_1+2h_2 t_k+3h_3 t_k^2=V_{3B};$$

$$\rho_1+2\rho_2 t_k+3\rho_3 t_k^2=V_{2B}.$$

It is quite obvious that

$$\omega=\frac{V_{2A}}{r_{1A}}\quad\text{or}\quad\omega=\frac{V_{1B}}{r_{2B}}\quad\text{that is}\quad\frac{V_{2A}}{V_{1B}}=\frac{r_{1A}}{r_{2B}},$$

and $t_k=\dfrac{\pi}{2\omega}$ or $t_k=\dfrac{\pi r_{1A}}{2V_{2A}}$, $t_k=\dfrac{\pi r_{2B}}{2V_{1B}}$.

Thus, the following systems of equations are independent

$$\begin{cases} t_k^2\left(\rho_2+\rho_3 t_k\right)=r_{2B}-r_{1A}-V_{1A}\dfrac{\pi r_{1A}}{2V_{2A}};\\ t_k\left(2\rho_2+3\rho_3 t_k\right)=V_{2B}-V_{1A} \end{cases}$$

$$\begin{cases} t_k^2\left(h_2+h_3 t_k\right)=r_{3B}-r_{3A}-V_{3A}\dfrac{\pi r_{1A}}{2V_{2A}}.\\ t_k\left(2h_2+3h_3 t_k\right)=V_{3B}-V_{3A} \end{cases}$$

And their solution is

$$\rho_2=3\frac{r_{2B}-r_{1A}}{t_k^{\,2}}-\frac{V_{2B}+2V_{1A}}{t_k};$$

$$\rho_3 = -2\frac{r_{2B} - r_{1A}}{t_k^3} - \frac{V_{2B} + V_{1A}}{t_k^2};$$

$$h_2 = 3\frac{r_{3B} - r_{3A}}{t_k^2} - \frac{V_{3B} + 2V_{3A}}{t_k};$$

$$h_3 = -2\frac{r_{3B} - r_{3A}}{t_k^3} + \frac{V_{3B} + V_{3A}}{t_k^2},$$

or while expressing the time required for a vehicle to cross the set route section t_k through kinematic initial data within the boundary points of the section in the formulas, we obtain

$$\rho_2 = \frac{12}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2 \left(r_{2B} - r_{1A}\right) - \frac{2}{\pi}\frac{V_{2A}}{r_{1A}}\left(V_{2B} + 2V_{1A}\right);$$

$$\rho_3 = -\frac{16}{\pi^3}\left(\frac{V_{2A}}{r_{1A}}\right)^3 \left(r_{2B} - r_{1A}\right) + \frac{4}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2 \left(V_{2B} + 2V_{1A}\right);$$

$$h_2 = \frac{12}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2 \left(r_{3B} - r_{3A}\right) - \frac{2}{\pi}\frac{V_{2A}}{r_{1A}}\left(V_{3B} + 2V_{3A}\right);$$

$$h_3 = -\frac{16}{\pi^3}\left(\frac{V_{2A}}{r_{1A}}\right)^3 \left(r_{3B} - r_{3A}\right) + \frac{4}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2 \left(V_{3B} + 2V_{3A}\right).$$

Thus, the hodograph of ascending (descending) vehicle motion and its rotation at a right angle in terms of spatial route is determined as follows

$$\bar{r}(t) = \left[r_{1A} + V_{1A}t + \frac{2}{\pi}\frac{V_{2A}}{r_{1A}}\left(\frac{6}{\pi}\frac{V_{2A}}{r_{1A}}\left(r_{2B} - r_{1A}\right) - V_{2B} - 2V_{1A}\right)t^2 + \right.$$
$$\left. + \frac{4}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2 \left(V_{2B} + V_{1A} - \frac{4}{\pi}\frac{V_{2A}}{r_{1A}}\left(r_{2B} - r_{1A}\right)\right)t^3 \right] \times$$
$$\times \left(\bar{i}\cos\left(\frac{V_{2A}}{r_{1A}}t\right) + \bar{j}\sin\left(\frac{V_{2A}}{r_{1A}}t\right)\right) +$$
$$+ \left[r_{3A} + V_{3A}t + \frac{2}{\pi}\frac{V_{2A}}{r_{1A}}\left(\frac{6}{\pi}\frac{V_{2A}}{r_{1A}}\left(r_{3B} - r_{3A}\right) - V_{3B} - 2V_{3A}\right)t^2 + \right.$$
$$\left. + \frac{4}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2 \left(V_{3B} + V_{3A} - \frac{4}{\pi}\frac{V_{2A}}{r_{1A}}\left(r_{3B} - r_{3A}\right)\right)t^3 \right] \cdot \bar{k}.$$

More specifically, when $V_{1A} = 0$, $V_{3A} = 0$, $V_{2B} = 0$, $V_{3B} = 0$, we obtain

$$\bar{r}(t) = \left[r_{1A} + \frac{12}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2 \left(r_{2B} - r_{1A}\right)t^2 - \right.$$
$$\left. - \frac{16}{\pi^3}\left(\frac{V_{2A}}{r_{1A}}\right)^3 \left(r_{2B} - r_{1A}\right)t^3 \right]\left(\bar{i}\cos\left(\frac{V_{2A}}{r_{1A}}t\right) + \bar{j}\sin\left(\frac{V_{2A}}{r_{1A}}t\right)\right) +$$
$$+ \left[r_{3A} + \frac{12}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2 \left(r_{3B} - r_{3A}\right)t^2 - \frac{16}{\pi^3}\left(\frac{V_{2A}}{r_{1A}}\right)^3 \left(r_{3B} - r_{3A}\right)t^3 \right] \cdot \bar{k}.$$

Further, while considering that $r_{3A} = 0$ and $r_{3B} = 0$, we obtain a hodograph of a vehicle movement in terms of straight angle rotation within the horizontal surface.

Natural trihedral of the supporting trajectory. The set program hodograph of a vehicle transfer should be used to determine orientation of natural trihedral of spiral-helix supporting trajectory in terms of earth reference.

Fig. 2 demonstrates $\bar{r}(t)$ hodograph of program motion $\breve{AB}$ of a vehicle in terms of the fixed earth reference and natural trihedral of the supporting trajectory connected with variable material point *M*.

In this context, $\bar{\tau}, \bar{n}, \bar{b}$ are orts of the natural trihedral of the supporting trajectory.

It is known that unit vectors of a natural trihedral of the spatial curve are determined by the formulas

$$\bar{\tau} = \frac{d\bar{r}}{ds}; \quad \bar{n} = \frac{1}{K}\frac{d^2\bar{r}}{ds^2}; \quad \bar{b} = \bar{\tau} \times \bar{n},$$

where *s* is arc the length of the spatial curve, and *K* is the arc.

In this context, the arc is determined in the form of

$$K = \left|\frac{d\bar{r}}{ds}\right|.$$

Since the hodograph of a program motion is set as the vector function of time $\bar{r}(t)$, it is expedient to reconstruct the formulas of natural trihedral orts in such a way

$$\bar{\tau} = \frac{1}{\left(\dot{\bar{r}} \cdot \dot{\bar{r}}\right)^{\frac{1}{2}}}\dot{\bar{r}}; \quad \bar{n} = -\frac{1}{\left(\dot{\bar{r}} \cdot \dot{\bar{r}}\right)^{\frac{1}{2}}\left|\dot{\bar{r}} \times \ddot{\bar{r}}\right|}\dot{\bar{r}} \times \left(\dot{\bar{r}} \times \ddot{\bar{r}}\right);$$

$$\bar{b} = -\frac{1}{\left(\dot{\bar{r}} \cdot \dot{\bar{r}}\right)\left|\dot{\bar{r}} \times \ddot{\bar{r}}\right|}\dot{\bar{r}} \times \left[\dot{\bar{r}} \times \left(\dot{\bar{r}} \times \ddot{\bar{r}}\right)\right],$$

where $\left|\dot{\bar{r}}\right| = \sqrt{\left(\dot{\bar{r}} \cdot \dot{\bar{r}}\right)}$; $\dot{\bar{r}}$, $\ddot{\bar{r}}$ are the first and second order time derivatives of radius-vector (program hodograph) respectively.

Let us note that owing to the identical equations we have [5]

$$\bar{a} \times \left(\bar{b} \times \bar{c}\right) = \begin{vmatrix} \bar{b} & \bar{c} \\ \bar{a} \cdot \bar{b} & \bar{a} \cdot \bar{c} \end{vmatrix};$$

$$\bar{a} \times \left[\bar{b} \times \left(\bar{c} \times \bar{d}\right)\right] = \begin{vmatrix} \bar{a} \times \bar{c} & \bar{a} \times \bar{d} \\ \bar{b} \cdot \bar{c} & \bar{b} \cdot \bar{d} \end{vmatrix}.$$

We also obtain

$$\bar{n} = -\frac{1}{\left|\dot{\bar{r}}\right|\left|\dot{\bar{r}} \times \ddot{\bar{r}}\right|}\left[\left(\dot{\bar{r}} \cdot \dot{\bar{r}}\right)\ddot{\bar{r}} - \left(\dot{\bar{r}} \cdot \ddot{\bar{r}}\right)\dot{\bar{r}}\right];$$

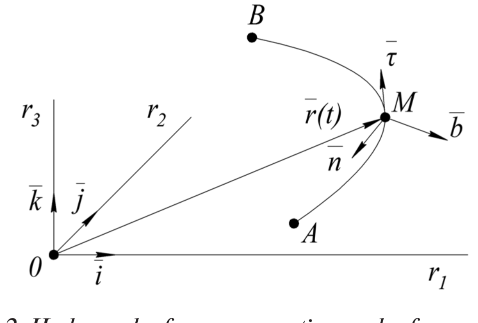

Fig. 2. Hodograph of program motion and reference system

$$\bar{b} = \frac{1}{\left|\dot{\bar{r}} \times \ddot{\bar{r}}\right|}\left(\dot{\bar{r}} \times \ddot{\bar{r}}\right).$$

Let us represent the vector operations in the coordinate form applying mathematical apparatus of quaternion matrices [5]

$$\dot{\bar{r}} \times \ddot{\bar{r}} \to \frac{1}{2}\left(\dot{R}_0 - \dot{R}_0^t\right)\ddot{\bar{r}};$$

$$\dot{\bar{r}} \times \left(\dot{\bar{r}} \times \ddot{\bar{r}}\right) \to \frac{1}{4}\left(\dot{R}_0 - \dot{R}_0^t\right)^2 \ddot{\bar{r}};$$

$$\dot{\bar{r}} \times \left[\dot{\bar{r}} \times \left(\dot{\bar{r}} \times \ddot{\bar{r}}\right)\right] \to \frac{1}{8}\left(\dot{R}_0 - \dot{R}_0^t\right)^3 \ddot{\bar{r}},$$

where

$$\dot{R}_0 = \begin{Vmatrix} 0 & \dot{r}_1 & \dot{r}_2 & \dot{r}_3 \\ -\dot{r}_1 & 0 & -\dot{r}_3 & \dot{r}_2 \\ -\dot{r}_2 & \dot{r}_3 & 0 & -\dot{r}_1 \\ -\dot{r}_3 & -\dot{r}_2 & \dot{r}_1 & 0 \end{Vmatrix};$$

$$\dot{R}_0^t = \begin{Vmatrix} 0 & \dot{r}_1 & \dot{r}_2 & \dot{r}_3 \\ -\dot{r}_1 & 0 & \dot{r}_3 & -\dot{r}_2 \\ -\dot{r}_2 & -\dot{r}_3 & 0 & \dot{r}_1 \\ -\dot{r}_3 & \dot{r}_2 & -\dot{r}_1 & 0 \end{Vmatrix}.$$

Expansion of the equations helps find the following

$$\frac{1}{2}\left(\dot{R}_0 - \dot{R}_0^t\right)\ddot{\bar{r}} = \frac{1}{2} \cdot 2 \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\dot{r}_3 & \dot{r}_2 \\ 0 & \dot{r}_3 & 0 & -\dot{r}_1 \\ 0_3 & -\dot{r}_2 & \dot{r}_1 & 0 \end{Vmatrix} \begin{Vmatrix} 0 \\ \ddot{r}_1 \\ \ddot{r}_2 \\ \ddot{r}_3 \end{Vmatrix} = \begin{Vmatrix} 0 \\ \dot{r}_2\ddot{r}_3 - \dot{r}_3\ddot{r}_2 \\ \dot{r}_3\ddot{r}_1 - \dot{r}_1\ddot{r}_3 \\ \dot{r}_1\ddot{r}_2 - \dot{r}_2\ddot{r}_1 \end{Vmatrix};$$

$$\frac{1}{4}\left(\dot{R}_0 - \dot{R}_0^t\right)^2 \ddot{\bar{r}} = \frac{1}{4} \cdot \left(2 \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\dot{r}_3 & \dot{r}_2 \\ 0 & \dot{r}_3 & 0 & -\dot{r}_1 \\ 0_3 & -\dot{r}_2 & \dot{r}_1 & 0 \end{Vmatrix} \right)^2 \begin{Vmatrix} 0 \\ \ddot{r}_1 \\ \ddot{r}_2 \\ \ddot{r}_3 \end{Vmatrix} =$$

$$= \begin{Vmatrix} 0 \\ \dot{r}_1\left(\dot{r}_2\ddot{r}_2 + \dot{r}_3\ddot{r}_3\right) - \ddot{r}_1\left(\dot{r}_3\dot{r}_3 + \dot{r}_2\dot{r}_2\right) \\ \dot{r}_2\left(\dot{r}_1\ddot{r}_1 + \dot{r}_3\ddot{r}_3\right) - \ddot{r}_2\left(\dot{r}_3\dot{r}_3 + \dot{r}_1\dot{r}_1\right) \\ \dot{r}_3\left(\dot{r}_1\ddot{r}_1 + \dot{r}_2\ddot{r}_2\right) - \ddot{r}_3\left(\dot{r}_2\dot{r}_2 + \dot{r}_1\dot{r}_1\right) \end{Vmatrix};$$

$$\frac{1}{8}\left(\dot{R}_0 - \dot{R}_0^t\right)^3 \ddot{\bar{r}} = \frac{1}{8} \cdot \left(2 \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\dot{r}_3 & \dot{r}_2 \\ 0 & \dot{r}_3 & 0 & -\dot{r}_1 \\ 0_3 & -\dot{r}_2 & \dot{r}_1 & 0 \end{Vmatrix} \right)^3 \begin{Vmatrix} 0 \\ \ddot{r}_1 \\ \ddot{r}_2 \\ \ddot{r}_3 \end{Vmatrix} =$$

$$= \left(\dot{\bar{r}} \cdot \dot{\bar{r}}\right) \begin{Vmatrix} 0 \\ \dot{r}_3\ddot{r}_2 - \dot{r}_2\ddot{r}_3 \\ \dot{r}_1\ddot{r}_3 - \dot{r}_3\ddot{r}_1 \\ \dot{r}_2\ddot{r}_1 - \dot{r}_1\ddot{r}_2 \end{Vmatrix}.$$

Thus, in terms of the inertial reference system, the following column vectors correspond to the natural trihedral orts

$$\bar{\tau} = \frac{1}{\left|\dot{\bar{r}}\right|} \begin{Vmatrix} 0 \\ \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{Vmatrix};$$

$$\bar{n} = -\frac{1}{\left|\dot{\bar{r}}\right|\left|\dot{\bar{r}} \times \ddot{\bar{r}}\right|} \begin{Vmatrix} 0 \\ \ddot{r}_1\left(\dot{r}_3\dot{r}_3 + \dot{r}_2\dot{r}_2\right) - \dot{r}_1\left(\dot{r}_2\ddot{r}_2 + \dot{r}_3\ddot{r}_3\right) \\ \ddot{r}_2\left(\dot{r}_3\dot{r}_3 + \dot{r}_1\dot{r}_1\right) - \dot{r}_2\left(\dot{r}_1\ddot{r}_1 + \dot{r}_3\ddot{r}_3\right) \\ \ddot{r}_3\left(\dot{r}_2\dot{r}_2 + \dot{r}_1\dot{r}_1\right) - \dot{r}_3\left(\dot{r}_1\ddot{r}_1 + \dot{r}_2\ddot{r}_2\right) \end{Vmatrix};$$

$$\bar{b} = \frac{1}{\left|\dot{\bar{r}} \times \ddot{\bar{r}}\right|} \begin{Vmatrix} 0 \\ \dot{r}_2\ddot{r}_3 - \dot{r}_3\ddot{r}_2 \\ \dot{r}_3\ddot{r}_1 - \dot{r}_1\ddot{r}_3 \\ \dot{r}_1\ddot{r}_2 - \dot{r}_2\ddot{r}_1 \end{Vmatrix},$$

or

$$\bar{\tau} = \frac{1}{\left|\dot{\bar{r}}\right|}\left[\dot{r}_1\bar{i} + \dot{r}_2\bar{j} + \dot{r}_3\bar{k}\right];$$

$$\bar{n} = \frac{1}{\left|\dot{\bar{r}}\right|\left|\dot{\bar{r}} \times \ddot{\bar{r}}\right|} \left[\begin{vmatrix} \ddot{r}_1 & \dot{r}_2\ddot{r}_2 + \dot{r}_3\ddot{r}_3 \\ \dot{r}_1 & \dot{r}_2\dot{r}_2 + \dot{r}_3\dot{r}_3 \end{vmatrix} \bar{i} + \begin{vmatrix} \ddot{r}_2 & \dot{r}_1\ddot{r}_1 + \dot{r}_3\ddot{r}_3 \\ \dot{r}_2 & \dot{r}_1\dot{r}_1 + \dot{r}_3\dot{r}_3 \end{vmatrix} \bar{j} + \right.$$
$$\left. + \begin{vmatrix} \ddot{r}_3 & \dot{r}_1\ddot{r}_1 + \dot{r}_2\ddot{r}_2 \\ \dot{r}_3 & \dot{r}_1\dot{r}_1 + \dot{r}_2\dot{r}_2 \end{vmatrix} \bar{k} \right];$$

$$\bar{b} = \frac{1}{\left|\dot{\bar{r}} \times \ddot{\bar{r}}\right|} \left[\begin{vmatrix} \dot{r}_2 & \dot{r}_3 \\ \ddot{r}_2 & \ddot{r}_3 \end{vmatrix} \bar{i} + \begin{vmatrix} \dot{r}_3 & \dot{r}_1 \\ \ddot{r}_3 & \ddot{r}_1 \end{vmatrix} \bar{j} + \begin{vmatrix} \dot{r}_1 & \dot{r}_2 \\ \ddot{r}_1 & \ddot{r}_2 \end{vmatrix} \bar{k} \right].$$

Orientation of the natural trihedral on the matrix of directional cosines. In the general case, the relative rotation of two coordinate systems is determined with the help of a matrix of directional cosines of the following type

$$\begin{Vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{Vmatrix}.$$

The following transformation matrix determines the natural trihedral orientation within the inertial space for random moment of movement along the program trajectory

$$\begin{Vmatrix} \frac{1}{\left|\dot{\bar{r}}\right|}\dot{r}_1 & \frac{1}{\left|\dot{\bar{r}}\right|}\dot{r}_2 & \frac{1}{\left|\dot{\bar{r}}\right|}\dot{r}_3 \\ \frac{1}{\left|\dot{\bar{r}}\right|\left|\dot{\bar{r}} \times \ddot{\bar{r}}\right|}\begin{vmatrix} \ddot{r}_1 & \dot{r}_2\ddot{r}_2 + \dot{r}_3\ddot{r}_3 \\ \dot{r}_1 & \dot{r}_2\dot{r}_2 + \dot{r}_3\dot{r}_3 \end{vmatrix} & \frac{1}{\left|\dot{\bar{r}}\right|\left|\dot{\bar{r}} \times \ddot{\bar{r}}\right|}\begin{vmatrix} \ddot{r}_2 & \dot{r}_1\ddot{r}_1 + \dot{r}_3\ddot{r}_3 \\ \dot{r}_2 & \dot{r}_1\dot{r}_1 + \dot{r}_3\dot{r}_3 \end{vmatrix} & \frac{1}{\left|\dot{\bar{r}}\right|\left|\dot{\bar{r}} \times \ddot{\bar{r}}\right|}\begin{vmatrix} \ddot{r}_3 & \dot{r}_1\ddot{r}_1 + \dot{r}_2\ddot{r}_2 \\ \dot{r}_3 & \dot{r}_1\dot{r}_1 + \dot{r}_2\dot{r}_2 \end{vmatrix} \\ \frac{1}{\left|\dot{\bar{r}} \times \ddot{\bar{r}}\right|}\begin{vmatrix} \dot{r}_2 & \dot{r}_3 \\ \ddot{r}_2 & \ddot{r}_3 \end{vmatrix} & \frac{1}{\left|\dot{\bar{r}} \times \ddot{\bar{r}}\right|}\begin{vmatrix} \dot{r}_3 & \dot{r}_1 \\ \ddot{r}_3 & \ddot{r}_1 \end{vmatrix} & \frac{1}{\left|\dot{\bar{r}} \times \ddot{\bar{r}}\right|}\begin{vmatrix} \dot{r}_1 & \dot{r}_2 \\ \ddot{r}_1 & \ddot{r}_2 \end{vmatrix} \end{Vmatrix}.$$

Thus, putting together the matrices, we obtain

$$\alpha_{11} = \frac{1}{\left|\dot{\bar{r}}\right|}\dot{r}_1;$$

$$\alpha_{12} = \frac{1}{\left|\dot{\bar{r}}\right|}\dot{r}_2;$$

$$\alpha_{13}=\frac{1}{\left|\dot{\bar{r}}\right|}\dot{r}_3;$$

$$\alpha_{21}=\frac{1}{\left|\dot{\bar{r}}\right|\left|\dot{\bar{r}}\times\ddot{\bar{r}}\right|}\begin{vmatrix}\ddot{r}_1 & \dot{r}_2\ddot{r}_2+\dot{r}_3\ddot{r}_3\\ \dot{r}_1 & \dot{r}_2\dot{r}_2+\dot{r}_3\dot{r}_3\end{vmatrix};$$

$$\alpha_{22}=\frac{1}{\left|\dot{\bar{r}}\right|\left|\dot{\bar{r}}\times\ddot{\bar{r}}\right|}\begin{vmatrix}\ddot{r}_2 & \dot{r}_1\ddot{r}_1+\dot{r}_3\ddot{r}_3\\ \dot{r}_2 & \dot{r}_1\dot{r}_1+\dot{r}_3\dot{r}_3\end{vmatrix};$$

$$\alpha_{23}=\frac{1}{\left|\dot{\bar{r}}\right|\left|\dot{\bar{r}}\times\ddot{\bar{r}}\right|}\begin{vmatrix}\ddot{r}_3 & \dot{r}_1\ddot{r}_1+\dot{r}_2\ddot{r}_2\\ \dot{r}_3 & \dot{r}_1\dot{r}_1+\dot{r}_2\dot{r}_2\end{vmatrix};$$

$$\alpha_{31}=\frac{1}{\left|\dot{\bar{r}}\times\ddot{\bar{r}}\right|}\begin{vmatrix}\dot{r}_2 & \dot{r}_3\\ \ddot{r}_2 & \ddot{r}_3\end{vmatrix};$$

$$\alpha_{32}=\frac{1}{\left|\dot{\bar{r}}\times\ddot{\bar{r}}\right|}\begin{vmatrix}\dot{r}_3 & \dot{r}_1\\ \ddot{r}_3 & \ddot{r}_1\end{vmatrix};$$

$$\alpha_{33}=\frac{1}{\left|\dot{\bar{r}}\times\ddot{\bar{r}}\right|}\begin{vmatrix}\dot{r}_1 & \dot{r}_2\\ \ddot{r}_1 & \ddot{r}_2\end{vmatrix}.$$

Components of the matrix of the directional cosines have been identified by means of the first- and second-time derivatives on the components of a radius vector of the spiral-helix program trajectory

$$\dot{r}_1=\left\|\rho_0\rho_1\rho_2\rho_3\right\|\left[\begin{Vmatrix}0\\1\\2t\\3t^2\end{Vmatrix}\cos\omega t-\omega\begin{Vmatrix}1\\t\\t^2\\t^3\end{Vmatrix}\sin\omega t\right];$$

$$\dot{r}_2=\left\|\rho_0\rho_1\rho_2\rho_3\right\|\left[\begin{Vmatrix}0\\1\\2t\\3t^2\end{Vmatrix}\sin\omega t+\omega\begin{Vmatrix}1\\t\\t^2\\t^3\end{Vmatrix}\cos\omega t\right];$$

$$\dot{r}_3=\left\|h_0h_1h_2h_3\right\|\begin{Vmatrix}0\\1\\2t\\3t^2\end{Vmatrix};$$

$$\ddot{r}_1=\left\|\rho_0\rho_1\rho_2\rho_3\right\|\left[\begin{Vmatrix}-\omega^2\\-\omega^2t\\2-\omega^2t^2\\6t-\omega^2t^2\end{Vmatrix}\cos\omega t-2\omega\begin{Vmatrix}0\\1\\2t\\3t^2\end{Vmatrix}\sin\omega t\right];$$

$$\ddot{r}_2=\left\|\rho_0\rho_1\rho_2\rho_3\right\|\left[\begin{Vmatrix}-\omega^2\\-\omega^2t\\2-\omega^2t^2\\6t-\omega^2t^2\end{Vmatrix}\sin\omega t-2\omega\begin{Vmatrix}0\\1\\2t\\3t^2\end{Vmatrix}\cos\omega t\right];$$

$$\ddot{r}_3=\left\|h_0h_1h_2h_3\right\|\begin{Vmatrix}0\\0\\2\\6t\end{Vmatrix}.$$

Orientation of the natural trihedral on Rodriguez-Hamilton parameters. In terms of Rodriguez-Hamilton parameters, spatial rotation is determined by means of quaternion matrices in the form of [5]

$$A^t\cdot{}^tA^t.$$

In this context, the quaternion matrices

$$A^t=\begin{Vmatrix}\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3\\ -\alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2\\ -\alpha_2 & -\alpha_3 & \alpha_0 & \alpha_1\\ -\alpha_3 & \alpha_2 & -\alpha_1 & \alpha_0\end{Vmatrix};$$

$${}^tA^t=\begin{Vmatrix}\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3\\ \alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2\\ \alpha_2 & -\alpha_3 & \alpha_0 & \alpha_1\\ \alpha_3 & \alpha_2 & -\alpha_1 & \alpha_0\end{Vmatrix},$$

follow Rodriguez-Hamilton parameters: a_0, a_1, a_2, a_3. The required Rodriguez-Hamilton parameters determining the natural trihedral orientation on the prescribed program motion hodograph may be identified directly from the system of four independent algebraic equations

$$\begin{cases}\alpha_0^2+\alpha_1^2+\alpha_2^2+\alpha_3^2=1\\ \alpha_0^2+\alpha_1^2-\alpha_2^2-\alpha_3^2=\alpha_{11}\\ \alpha_0^2-\alpha_1^2+\alpha_2^2-\alpha_3^2=\alpha_{22}\\ \alpha_0^2-\alpha_1^2-\alpha_2^2+\alpha_3^2=\alpha_{33}\end{cases},$$

or in terms of a matrix form

$$\begin{Vmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{Vmatrix}\begin{Vmatrix}\alpha_0^2\\ \alpha_1^2\\ \alpha_2^2\\ \alpha_3^2\end{Vmatrix}=\begin{Vmatrix}1\\ \alpha_{11}\\ \alpha_{22}\\ \alpha_{33}\end{Vmatrix}.$$

It should be noted that the matrix

$$\begin{Vmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{Vmatrix},$$

is a symmetrical, nondegenerate matrix having the following property of orthogonality

$$\begin{Vmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{Vmatrix}\cdot\begin{Vmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{Vmatrix}=4\begin{Vmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{Vmatrix}.$$

Thus, the inverse matrix is

$$\frac{1}{4}\begin{Vmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{Vmatrix}.$$

Hence, the solution may be found in the following matrix form

$$\begin{Vmatrix}\alpha_0^2\\ \alpha_1^2\\ \alpha_2^2\\ \alpha_3^2\end{Vmatrix}=\frac{1}{4}\begin{Vmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{Vmatrix}\begin{Vmatrix}1\\ \alpha_{11}\\ \alpha_{22}\\ \alpha_{33}\end{Vmatrix}.$$

Natural trihedral orientation on Gibbs vector. Spatial rotation is determined univalently according to Gibbs vector. Components of Gibbs vector are invariant and fixed in terms of dynamical frame of reference and fixed one, i. e.

$$\begin{Vmatrix}G_1 & G_2 & G_3\end{Vmatrix}\begin{Vmatrix}\bar{\tau}\\ \bar{n}\\ \bar{b}\end{Vmatrix}=\begin{Vmatrix}G_1 & G_2 & G_3\end{Vmatrix}\begin{Vmatrix}\bar{i}\\ \bar{j}\\ \bar{k}\end{Vmatrix}.$$

Components of Gibbs vector are connected with Rodriguez-Hamilton parameters by means of the known formulas

$$G_1=\frac{\alpha_1}{\alpha_0};$$

$$G_2=\frac{\alpha_2}{\alpha_0};$$

$$G_3=\frac{\alpha_3}{\alpha_0}.$$

Components of Gibbs vector are determined using the components of matrices of directional cosines with the help of the following identical forms

$$G_1=\frac{\alpha_{12}+\alpha_{21}}{\alpha_{31}-\alpha_{13}}=\frac{\alpha_{13}+\alpha_{31}}{\alpha_{12}-\alpha_{21}};$$

$$G_2=\frac{\alpha_{12}+\alpha_{21}}{\alpha_{23}-\alpha_{32}}=\frac{\alpha_{23}+\alpha_{32}}{\alpha_{12}-\alpha_{21}};$$

$$G_3=\frac{\alpha_{13}+\alpha_{31}}{\alpha_{23}-\alpha_{32}}=\frac{\alpha_{23}+\alpha_{32}}{\alpha_{31}-\alpha_{13}}.$$

Conclusions. A hodograph of a vehicle program transition in terms of earth reference system has been proposed in the class of spiral-helix supporting trajectories. Unit vectors of the natural trihedral of spiral-helix supporting trajectory have been obtained depending upon the time derivatives of the program motion hodograph. Calculation algorithms for complicated operations of vector algebra in terms of quaternion matrices are demonstrated. The components of Gibbs vector, Rodriguez-Hamilton parameters, and a matrix of directional cosines have been determined as the functions of kinematic parameters of the program motion. The program transition – radius-vector hodograph and program rotation (orientation) – Gibbs vector or a quaternion on Rodriguez-Hamilton parameters has been represented vectorially. This makes it possible to model spatial problems concerning the orientation and control of dynamic systems (vehicles) while using mathematical apparatus of quaternion matrices being practical for computer facilities.

Involving the unified position of mathematical apparatus of vector algebra, the vector approach to the description of a program spatial transition and orientation helps formulate and solve nonlinear problems of a vehicle dynamics, find the required contact driving forces providing the controllable program motion along the supporting path. The estimation of program contact (controlling) driving forces opens up the possibility for the substantiated solution of problems of dynamic vehicle design involving the required power supply capacity, selection of inertial characteristics, geometry, structural schemes, aerodynamic shapes, and others.

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Орієнтація натурального тріедра спірально-гвинтової опорної траєкторії руху автомобіля у просторі

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Мета. Підвищення достовірності та інженерної точності 3D-моделювання руху автомобіля.

Методика. Орти натурального тріедра спірально-гвинтової опорної траєкторії надаються у векторній формі похідними за часом від радіуса-вектора програмного руху. Складні операції векторної алгебри наводяться кватерніонними матрицями. Орієнтація у просторі або програмний поворот натурального тріедра описується вектором Гіббса, параметрами Родріга-Гамільтона, матрицею напрямних косинусів у залежності від кінематичних параметрів програмного руху.

Результати. Годограф програмного перенесення автомобіля представлено у класі спірально-гвинтових опорних траєкторій у земній системі відліку. Одиничні вектори натурального тріедра спірально-гвинтової опорної траєкторії отримані в залежності від похідних за часом годографа програмного руху.

Наукова новизна. Програмне перенесення – годограф радіуса-вектора та програмний поворот (орієнтація) – вектор Гіббса або кватерніон за параметрами Родріга-Гамільтона представлені у векторній формі, що дозволяє моделювати просторові завдання орієнтації й керування динамічними системами (автомобілями) у кватерніонних матрицях.

Практична значимість. Розрахункові формули наводяться в упорядкованому, компактному матричному вигляді, безпосередньо адаптованому до комп'ютерних технологій. Алгоритм дозволяє вирішувати широке коло завдань динамічного проектування транспортних засобів.

Ключові слова: *годограф, спірально-гвинтова траєкторія, натуральний тріедр, орієнтація, вектор Гіббса, параметри Родріга-Гамільтона, кватерніонні матриці*

Ориентация натурального триэдра спирально-винтовой опорной траектории движения автомобиля в пространстве

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Цель. Повышение достоверности и инженерной точности 3D-моделирования движения автомобиля.

Методика. Орты натурального триэдра спирально-винтовой опорной траектории выражаются в векторной форме производными по времени от радиус-вектора программного движения. Сложные операции векторной алгебры представляются кватернионными матрицами. Ориентация в пространстве или программный поворот натурального триэдра описывается вектором Гиббса, параметрами Родрига-Гамильтона, матрицей направляющих косинусов в зависимости от кинематических параметров программного движения.

Результаты. Годограф программного переноса автомобиля представлен в классе спирально-винтовых опорных траекторий в земной системе отсчета. Единичные векторы натурального триэдра спирально-винтовой опорной траектории получены в зависимости от производных по времени годографа программного движения.

Научная новизна. Программный перенос – годограф радиус-вектора, и программный поворот (ориентация) – вектор Гиббса или кватернион по параметрам Родрига-Гамильтона, представлены в векторной форме, что позволяет моделировать пространственные задачи ориентации и управления динамическими системами (автомобилями) в кватернионных матрицах.

Практическая значимость. Расчетные формулы представлены в упорядоченном, компактном матричном виде, непосредственно адаптированном к компьютерным технологиям. Алгоритм позволяет решать широкий круг задач динамического проектирования транспортных средств.

Ключевые слова: *годограф, спирально-винтовая траектория, натуральный триэдр, ориентация, вектор Гиббса, параметры Родрига-Гамильтона, кватернионные матрицы*

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